A sample proof
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Theorem 1. The class of regular languages is contained in the class of languages recognized by NFAs.

Proof. Let $A$ be a regular language. Then there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $A = L(M)$.

We must show that $A$ is in the class of languages recognized by NFAs. That is, we must show that there exists an NFA that recognizes $A$. The proof of existence is by construction. We build an NFA $N = (Q, \Sigma, \delta_N, q_0, F)$, where the transition function $\delta_N : Q \times \Sigma \epsilon \rightarrow P(Q)$ is defined as follows.

$$\delta_N(q, \sigma) = \begin{cases} \emptyset & \text{if } \sigma = \epsilon \\ \{ \delta(q, \sigma) \} & \text{otherwise} \end{cases}$$

We now show correctness of our construction.

Claim 1. A string $w \in A$ if and only if $w \in L(N)$.

Proof. $(\Rightarrow)$ Assume $w = w_1w_2\ldots w_n \in A$, where $w_i \in \Sigma$ for all $i = 1, \ldots, n$. Then $w \in L(M)$, i.e., the DFA $M$ accepts input $w$. By definition, there exists a sequence of states $s_0, s_1, \ldots, s_n$ such that

1. $s_0 = q_0$
2. $s_{i+1} = \delta(s_i, w_{i+1})$ for all $i = 0, \ldots, n - 1$
3. $s_n \in F$

We show that $w \in L(N)$, i.e., that $N$ accepts $w$. First, note that $w$ can be written as $w_1w_2\ldots w_n$ with each $w_i \in \Sigma \subseteq \Sigma_e$ for all $i = 1, \ldots, n$. We verify that the sequence of states $s_0, \ldots, s_n$ is a valid computation path for $N$:

1. $s_0 = q_0$ by assumption
2. By definition of the transition function $\delta_N$, $\delta_N(s_i, w_{i+1}) = \{ \delta(s_i, w_{i+1}) \}$ since every $w_{i+1} \neq \epsilon$ by assumption for $i = 0, \ldots, n - 1$. Furthermore, since $\delta(s_i, w_{i+1}) = s_{i+1}$ by assumption, we have $\delta_N(s_i, w_{i+1}) = \{ s_{i+1} \}$. Then we satisfy that $s_{i+1} \in \delta_N(s_i, w_{i+1})$ for all $i = 0, \ldots, n - 1$. 

3. \( s_n \in F \) by assumption

\((\Leftarrow)\)

Assume \( w = w_1w_2 \ldots w_n \in L(N) \), where \( w_i \in \Sigma \) for all \( i = 1, \ldots, n \). Then the NFA \( N \) accepts input \( w \). Thus, we can write \( w = w_1w_2 \ldots w_m \) with each \( w_i \in \Sigma \), and there exists a sequence of states \( s_0, \ldots, s_m \) such that

1. \( s_0 = q_0 \)
2. \( s_{i+1} \in \delta_N(s_i, w_{i+1}) \) for all \( i = 0, \ldots, n - 1 \)
3. \( s_n \in F \)

We show that \( w \in A = L(M) \), i.e., that \( M \) accepts \( w \).

**Claim 2.** The string \( w = w_1w_2 \ldots w_m \) must be written with \( m = n \), i.e., with no extra \( \epsilon \) symbols.

**Proof.** The proof is by contradiction. Suppose there were some index \( j \) such that \( w_j = \epsilon \). By definition of the transition function \( \delta_N \), then, \( \delta_N(s_{j-1}, w_j) = \emptyset \). Then \( s_j \notin \delta_N(s_{j-1}, w_j) \), giving the contradiction. Thus, there can be no empty symbols and \( m = n \).

We now verify that the sequence of states \( s_0, \ldots, s_n \) is a valid computation path for the DFA \( M \).

1. \( s_0 = q_0 \) by assumption
2. Let \( i \in [0, \ldots, n - 1] \). By assumption, \( s_{i+1} \in \delta_N(s_i, w_{i+1}) \). Since \( w_{i+1} \neq \epsilon \) from the previous Claim, by the definition of the transition function \( \delta_N \), \( \delta_N(s_i, w_{i+1}) = \{ \delta(s_i, w_{i+1}) \} \). Since \( s_{i+1} \) is contained in this set of size exactly one by assumption, it must be the case that \( s_{i+1} = \delta(s_i, w_{i+1}) \).
3. \( s_n \in F \) by assumption

This concludes the proof of correctness.

Since our construction of the NFA \( N \) is correct, i.e., \( A = L(N) \), we have shown that the class of regular languages is contained in the class of languages recognized by NFAs.