Recall...

- Last time we showed that the class of regular languages is closed under:
  - Complement
  - Union
  - Intersection
Concatenation operation

- Let \( A \) and \( B \) be languages.
- The concatenation of \( A \) and \( B \) is
  \[ A \circ B = \{ xy \mid x \in A \text{ and } y \in B \}. \]

Example

- Let \( A = \{ w \mid w \text{ is a string of 0s and 1s containing an odd number of 1s} \} \)
- Let \( B = \{ w \mid w \text{ is a string of 0s and 1s containing an even number of 1s} \} \)
- What are \( A \circ B; B \circ A; A \circ A; B \circ B? \)
  - Are any of these languages regular?

Concatenation

- Conjecture: The class of regular languages is closed under the concatenation operation.
- Proof idea:
Hmm...

- $A = \{w \mid w$ is a string of 0s and 1s containing an odd number of 1s$\}$
- $B = \{w \mid w$ is a string of 0s and 1s containing an even number of 1s$\}$
- Create a machine by gluing... how?

We need another approach
The empty symbol $\varepsilon$

- Seems like it might work
- Let's try running it…

Relaxing the rules

**Deterministic (DFA)**

**Nondeterministic (NFA)**
Running DFA vs NFA

DFA computation (path) vs NFA computation (tree)

An example computation

- What about 1011?
Another example

- What language is being recognized?
- *Hint*: can you start listing strings accepted?

![Diagram]

Formally...

- A **nondeterministic finite automaton (NFA)** is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where
  - \(Q\) is a finite set called the *states*
  - \(\Sigma\) is a finite set called the *alphabet*
  - \(\delta: Q \times \Sigma \cup \{\varepsilon\} \rightarrow P(Q)\) is the *transition function*
  - \(q_0 \in Q\) is the *start state*
  - \(F \subseteq Q\) is a set of *accept states*

- In-class exercise:
NFA computation

• Let $N=(Q, \Sigma, \delta, q_0, F)$ be a NFA and let $w$ be a string over the alphabet $\Sigma$.

• Then $N$ accepts $w$ if
  
  - $w$ can be written as $w_1w_2w_3...w_m$ with each $w_i$ in $\Sigma^*$ and
  
  - There exists a sequence of states $s_0, s_1, s_2, ..., s_m$ exists in $Q$ with the following conditions:
    
    1. $s_0=q_0$
    2. $s_{i+1} \in \delta(s_i, w_{i+1})$ for $i = 0, ..., m-1$
    3. $s_m \in F$

Nondeterminism makes life easier

• Let’s build an NFA that recognizes $B = \{ w \mid w$ is a string over $\{a, b\}$ that starts and ends with the same symbol}\}$

$C = \{ w \mid w$ is a string over $\{0,1\}$ that contains at least three $1$’s\}$

$D = \{ w \mid w$ is a string over $\{0,1\}$ that contains at least three consecutive $1$’s\}$
If at first you don’t succeed...

... adjust your goal!

We wanted to prove:

The class of regular languages is closed under the concatenation operation.

If at first you don’t succeed...

... adjust your goal!

Instead, let’s prove the theorem:

The class of languages recognized by NFAs is closed under the concatenation operation.